THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE INTERESTS OF TEACHERS OF MATHEMATICS

Volume IV June, 1912 Number 4

CONTENTS

Published Quarterly by the

ASSOCIATION OF TEACHERS OF MATHEMATICS FOR THE MIDDLE STATES AND MARYLAND

LANCASTER, PA., SYRACUSE, N. Y.

NEW YORK, N. Y., PHILADELPHIA, PA.

Entered as second-class matter October 7, 1909, at the Post Office at Lancaster, Pa., under the Act of March 3, 1879.

THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

THE MATHEMATICS TEACHER is published quarterly—September, December, March and June—under the auspices of the Association of Teachers of Mathematics for the Middle States and Maryland.

The annual subscription price is $1.00; single copies, 35 cents.

Remittances should be sent to the Syracuse office by draft on New York, Express Order or Money Order, payable to The Mathematics Teacher.

THE MATHEMATICS TEACHER,
41 NORTH QUEEN STREET, LANCASTER, PA.
OF SYRACUSE, N. Y.

THE MATHEMATICS TEACHER

EDITED BY
W. H. METZLER

ASSOCIATED WITH
EUGENE R. SMITH MAURICE J. BABB

VOLUME IV JUNE, 1912 NUMBER 4

WHAT SHOULD BE THE ESSENTIALS OF A COURSE IN MATHEMATICS FOR ELEMENTARY SCHOOLS IN NEW YORK CITY?

By GEORGE B. GERMANN.

Our individual answers to this question are largely controlled by our individual opinions, in which school tradition, or vocational demands, or mere predilection enter as largely determining factors. As yet we are not in possession of a sufficiently large body of scientifically treated data dealing with mathematical courses of study which we may use as norms for the determination of our answer. However, opinions on a particular subject of this kind are still of value in so far as they are based on intelligent experience with the factors specifically involved.

Practical people will probably agree on the following:

1. No course of study can be perfect or final. It attempts to meet a certain condition in a changing environment as best human experience can devise.

2. A course of study ought not to be the expression of an individual opinion as to how these conditions of the changing environment are to be met.

3. A course of study should be the outgrowth of the experiences of many individuals who are intimately concerned with the conditions which the course attempts to satisfy.

4. A course of study should be formulated for the purpose of satisfying the conditions of a specific educational situation. This implies that we preserve our poise and our balance; that

we perceive clearly the functioning values of the various factors that enter into the situation.

There are three commonly accepted principles underlying the formulation of a course of study:

I. The nature of the material selected is conditioned by the particular environment in which the pupils happen to be placed.

II. The selection of the material of instruction is conditioned partly by the future needs of the pupils, and partly by the mental abilities of the pupils at their several stages of development.

III. The arrangement of the material is conditioned by the capacities of the pupils; whence, as a corollary, arises the principle of arrangement with a view to the production of efficient results. I wish to consider each of these principles briefly in connection with the question propounded in the title of this paper.

I. The nature of the material from which we may select topics for a course and syllabus in mathematics for the elementary schools of New York City is well defined. We are in the midst of a vast center of commercial, manufacturing, mercantile and financial interests. These bring us into contact with the world at large at many points. The nature of the mathematical problems seems thus to be clearly indicated. Among the elements of this complex we find such matters as foreign exchange, domestic exchange, stocks and bonds, duties, taxes, brokerage, insurance of all kinds, industrial conditions that have quantitative relationships, agricultural problems that indirectly concern the city, and so one through a large list. And as a foundation for all this we have, as basic material, the fundamental operations with integers, with common fractions, and with decimals, the process of percentage, and the common measures in ordinary use.

II. But the regulative function of the principle of selection sharply limits the range of the field that may profitably be covered in the 850 hours allotted to mathematics during the eight years of our elementary school course. Hence arises the practical question, Out of all the interesting and possible facts and factors of our immediate environment that possess quantitative relationships, what elements may we safely and reasonably select as fit subject-matter of instruction for children ranging in age

from 6 years through 14 or 15 years? As previously indicated, our controlling aim in selection has reference to two important conditions:

1. The satisfaction of the practical needs of the pupils to be fulfilled by a particular subject;

2. Furnishing material for suitable mental reactions, whence will arise valuable intellectual habits that are fostered by the functioning of such study. With respect to the intellectual habits fostered by the study of elementary mathematics in our schools, I presume that we shall agree on the two following as of greatest importance: (*a*) The ability to comprehend simple quantitative relationships; (*b*) The ability to resolve those relationships into a simpler form with facility and reasonable accuracy.

Unfortunately for the theorist, the controlling aim and its conditions can not be applied to our situation with the mathematical precision of a formula. It happens that certain practical considerations of a deterrent rather than regulative character must needs inhibit any tendency to select material of instruction in accordance with this aim and its conditions pure and simple.

In the first place, we find confronting us the given time element in the schedule of studies. What we would select for the purpose of fostering mathematical abilities as they could, might, or ought to be fostered, is one proposition. What we ought to select in order to meet the requirements of a time schedule that formally allows for mathematics 125 minutes per week for the first year, 150 minutes per week for the second through the fifth year, and 200 minutes per week for the sixth through the eighth year, or a total of 850 hours for the entire course, is an entirely different proposition.

In the second place, we should keep clearly in view the deterrent function of the teaching factor. Mathematics is, for most people, a dry and uninteresting subject. The great majority of our elementary school teachers really know very little of mathematics beyond the simplest arithmetic. Their preparatory mathematical outlook has seldom extended beyond the twilight horizon of introductory álgebra and plane geometry, in the dim glimmer of which they once perhaps half discerned a few truths which have long since taken flight on the wings of forgetfulness.

Hence in the formulation of a course in mathematics we should remember that our teachers, while earnest and hard working and ambitious, are nevertheless not mathematical philosophers, nor are they philosophical mathematicians; for all of which we may, indeed, be truly thankful for the children's sake.

In the third place, we should not lose sight of the fact that a course of study formulated with the best of intentions, that a course perfectly practicable and educative under administrative conditions that effectively limit a class to 15 or 20 pupils who may be readily amenable to the ten commandments and the law dependent thereon, may be wholly visionary and futile when applied to our circumstances of administration. In our city schools we face registers of 50 and over, thus limiting opportunities for real teaching that begets power. Upon our administrative functions are imposed restrictions and limitations of a nature that cause to be retained in these overlarge classes the unfit, the wayward, the lazy, the indolent, and others of the submerged. These are plain facts, and in considering the making of a course of study we are obliged, as practical men, to face them.

In the fourth place, we find ourselves confronted by the present necessity of a uniform course of study for practically all elementary school pupils, with their multitude of individual differences, who are unregulated by a common ideal possible for adolescents, or for children of a common stock, of a common language, of a common tradition. This, therefore, postulates a course that shall be a feasible minimum requirement, with equal emphasis on "feasible" and on "minimum."

These four deterrent factors, viz., the time schedule, the teaching factor, the administrative problem referred to, and the present necessity for a uniform course of study for all sorts and conditions of children, are usually completely lost to sight by critics. But they are present, they are vital, and they make an insistent demand on our attention.

With these facts to control our ambitious fancies, I wish to present for your consideration the composite opinion of many teachers and principals relative to the topics that should be selected as a feasible and minimum basis for a course in mathematics for the elementary schools of New York City. This opinion is summarized in the following topics:

1. The four fundamental operations with integers.
2. The four fundamental operations with common fractions.
3. The four fundamental operations with decimal fractions.
4. Percentage as another aspect of fractions and decimals.
5. Very simple applications of percentage to real business situations in profit and loss, commercial discount, commission, taxes, duties, insurance, and simple interest; in all of which as a minimum requirement only the direct case is to be considered.
6. A working knowledge of the common measures in ordinary use in this city.

This general opinion is also in favor of the elimination of the following topics now directed to be taught in our elementary schools:

1. Algebra and inventional geometry.
2. The impossible mensuration now allotted to the 8A grade.
3. The metric system of weights and measures, for a working knowledge of which the science course may offer an opportunity.
4. The indirect cases in brokerage, taxes, commission, duties, insurance, commercial discount, and interest, seldom called for in even highly specialized business activities, and for which an understanding and a solution will be at hand when the business need therefor arises.
5. The ridiculous and time-consuming improbable topics in compound denominate numbers.
6. Longitude and time, which has its place in mathematical geography.
7. True discount.
8. A few odds and ends, such as the weight of potatoes, wheat, oats, etc., per bushel; troy weight; capacities of bins and cisterns; and the multiplication tables beyond 9 times.

In practice, many principals have been driven to the necessity of omitting various of these topics, though they zealously endeavor to have the entire course and syllabus taught as prescribed. With all of these topics eliminated, there would still remain much to be profitably taught that would be not only useful, but also highly educative as mathematical training. Indeed, it seems probable that a reduced course and syllabus would be much more useful and much more educative than the present overloaded and involved course. Our pupils would

have a chance to use what they acquire, for there would be time for wide application and for intensive drill of real worth; and they would have a chance to think about things mathematical, for they would be rid of the perplexing confusion that in a great measure is due to the presence in our course of traditional curiosities and of a multiplicity of topics.

As to the elimination of algebra, I wish to state that the elementary school principals who have been consulted are practically unanimously in favor of such a step. The opinions of the high school people were ascertained by means of a questionnaire issued last May in which I propounded the three following questions:

1. What is your frank opinion relative to the preparedness of our grammar school graduates in the algebra work now outlined for the $7B$ and $8A$ grades, when the graduates reach high school?

2. Does our grammar school algebra assist pupils to a better understanding of high school algebra?

3. Would you and your mathematics teachers recommend that the study of algebra be postponed until the first term in the high school?

Of the 19 high schools then established in our city, replies were received from 13. All of the replies to the first question may be summarized in the brief statement, "variable and inconsequential." In answer to the second question, 8 answers were positively in the negative; 2 stated that the pupils were slightly helped; 2 reported the mathematics department to be divided in opinion; and just one answered in the affirmative. The replies to the third question were more decisive: 10 answered in the affirmative; 1 answered in the negative with the qualification that the algebra time should not be taken from the time allotted to arithmetic; 1 reported negatively without qualification; and 1 reported a division of opinion in the mathematics department.

As to inventional geometry, I may state that it has recently been incorporated in the new course of study in drawing where it quite properly belongs.

III. The third of the principles underlying the formulation of a course of study has special reference to the arrangement of the material of instruction. Such arrangement is dependent on

the capacities of the pupils. The problem at this point is to grade the work with a twofold reference: (1) The matter should be arranged so as to accord with the abilities of the pupils; (2) the matter should be arranged so as to permit of the production of efficient results. It is in connection with this principle of arrangement that much hair-splitting arises, for here we trespass on the sacred domain of method. Shall we arrange the material spirally? Shall we revert to the older and substantial topical arrangement? Is a compromise preferable? The criticisms that come to me from intelligent critics who are daily laboring with our pupils, are unanimous on this one point; viz., that most of our children have mixed and hopelessly confused notions about various things mathematical, but definite and clearly defined notions about very few. Of course, there are all sorts of abilities, and even the same boy or girl develops in ability to comprehend mathematical relations as time goes on. Yet we are convinced that matters can be helped considerably by an arrangement of material that will give the teacher opportunity to develop the work more intensively than is now possible. Notions scattered over a large field, though we perceive the unity in diversity, do not appeal to the pupils in that philosophical way. The great majority of our pupils repeat glibly, perhaps, after us as we develop and interweave. But we all know how hopelessly they flounder when left alone. Too many topics and too many processes should not be crowded too thickly upon them—they are but children.

With these considerations in mind, I submit, as the result of my conferences with those who are in daily and intimate touch with the matters with which this paper deals, an arrangement of the subject-matter based approximately upon the following plan:

1st and 2d years: Much concrete work for the purpose of affording a basis for the fundamental operations with integers, and also for the purpose of fixing numerical facts. Limit the main work in the manipulation of numbers to the operations of addition and subtraction, within a gradually extended number range of four-place integers.

3d and 4th years: Development of the multiplication tables through 9's and their correlative division tables. Gradually apply these to the operations of multiplication first, and then

to division, the overlapping of these two important processes not to occur until reasonable facility in the former with one or two-place multipliers has been gained. To the end of the 4th year limit multipliers and divisors to three places. Gradually extend the number range to seven-place integers. Introduce simple fractional notions in connection with the measures studied, and in the 4th year limit the fraction work to finding the business fractional parts of three-place numbers that are multiples of the denominators. The work in addition and subtraction extended within the number range indicated for the grades.

5th year: The chief work of the two grades of this year limited to common fractions of a simple type, such as are to be met with in ordinary transactions; and to decimals of not more than six places.

6th year: Opportunity should be given for clinching the work in fractions and decimals, and for correlating with these the operations of percentage as applied to situations within the pupils' experiences or understanding.

7th year: Application of the processes of percentage to the business situations previously mentioned, including simple interest, requiring the use of the direct case only.

8th year: Survey of the entire course with special reference to certain underlying general principles of arithmetic as applied to integers, fractions, decimals, and percentage, thus also affording an opportunity to the schools to render pupils assistance along the lines of their special needs after graduating.

From this very brief and sketchy outline there has been omitted a number of topics, not purely operative in character, that furnish the raw materials with which the operations deal; such as, problem work, its nature and scope; measurements; comparisons; business forms; as to which the time at my disposal does not permit of more than the passing mention that they would naturally find a proper place and use in the several grades.

For the purpose of giving definiteness, unity, and coherence to a course of study, I commend to your consideration the plan of the present Philadelphia course of study in mathematics for elementary schools, which appeals to me as the best of the many that I have examined. In addition to a statement of work to

be done, there is printed in connection with each grade outline (1) clear and explicit directions for teaching the subject-matter, and (2) carefully selected type problems for regulating the degree of difficulty or of simplicity worth striving for. Something of this order of efficient control I suggest as worthy of consideration for the purpose of securing desirable homogeneity in the mathematical work of our schools.

To summarize: if I interpret the prevailing opinion of my colaborers correctly, the mathematical work of our elementary schools in New York City is in need of three distinct things:

1. Elimination of the traditional, the unnecessary, the too difficult.

2. Opportunity for an intensive treatment of topics, with due regard to reasonable correlation.

3. Suggestions as to the results that may reasonably be expected in the various grades.

PUBLIC SCHOOL 130,
BROOKLYN, N. Y.

ALGEBRA IN THE ELEMENTARY SCHOOLS.

By James H. Shipley.

I have never yet heard of a high school teacher's being satisfied with the first-year pupil's knowledge of common fractions, and the other day a I. A. girl frankly admitted that she could not multiply $12\frac{2}{3}$ by $15\frac{3}{4}$ because she had skipped one grade, had then had algebra and geometry, and the review didn't touch on mixed numbers. Of course, this was not the fault of any teacher, but the fault of a system which is trying to crowd too many things into too small a space. The real efficiency of the schools does not depend upon their being housed in million-dollar buildings, under a well-organized administrative force, excellent equipment, etc.; it depends upon two things,—what is taught, and how it is taught; and more especially upon the latter; for every teacher of mathematics knows that a pupil can derive as much permanent good from the study of a very few topics or theorems, so presented as to be pleasing to the pupil, or at least interesting, and at the same time make him think, as he can from ten times the amount of material "rammed home" with the sole object of being reproduced at examinations. The policy of standardizing everything by examinations is doing our expensive school system an untold injury; the report of the city superintendent compares the schools according to the number of their pupils who pass the examinations, and the principal warns the teacher that he is rated according to the number of his pupils that pass, and this pressure is passed on to the pupil. Until some method of close class-room observation and supervision is introduced with a view to allowing free rein to a teacher's individuality and originality even at the expense of his pupil's failing the conventional examinations, it is hardly worth while suggesting other changes.

And *what* is taught is not of paramount importance; a well-informed, broad-minded, ingenious teacher of history, who can instill patriotism into a pupil and impart to him a desire for further reading of history after he leaves school, can do him

more good, as a future citizen, than a poor teacher of algebra might; or a teacher of algebra, by presenting the subject in an intelligent and intelligible way so as not to overburden the pupil or put him under too much duress, but lead him to see some of the interesting features of the subject, can do him more good than a poor teacher of history, who only tries to make the pupil accumulate facts. And so with the mathematics in the elementary schools; if all teachers are obliged to cover more matter than they really have time for, no one can do anyone any good, and poor teachers and good teachers are reduced to about the same level. When a person stops to think of all the things that he has learned at one time or another, and then considers how much of it he remembers, he can realize of how little importance the subject matter really is. Of two boys, taught in two different ways, for two different reasons, one might say that "π is the ratio of the circumference of a circle to its diameter," and the other would say that he had measured the circumferences and diameters of several circles and always found the circumference to be a little more than three times as long as the diameter.

As many things as possible should be forced out of the elementary curriculum so that what remained might be studied more leisurely and thoroughly. For example, to try to teach music to some of the boys is a farce, for the boy "who hath not music in his soul" is more "fit for rapine, plunder, and murder" after the process than before; and about the same can be said of free-hand drawing and several other things. In place of these and geometry, could and should be introduced a course in elementary mechanical draughting. This is something in which 99 boys out of a 100 are interested; it teaches them to be neat and accurate, it interests them in geometric forms and prepares the way for formal geometry, and it is the beginning of a profession; for a good draughtsman is usually in demand.

As for pupils getting algebra in the elementary schools, there is no reason why they should not have it from the very start, and become as used to the algebraic methods as they are to the arithmetic, for the algebraic method is clearer and admits more readily of explanation; it would also admit a larger variety of verbal problems, even to the extent of two unknown quantities, which pupils find very easy. The fact of these problems being practical or not is immaterial—the effort to introduce so called

practical things into the curriculum is somewhat misdirected, for if a pupil in either the elementary or high schools can be taught enough English and arithmetic to see him through life, and in addition can be taught to think quickly and correctly and independently, he has acquired the most practical thing possible, and it makes little difference what he has studied to acquire this end, so long as he has been given an impulse to continue his work in literature or history or language or mathematics or biology, after he leaves school, as a recreation or pastime, free from compunction.

There are many ways in which algebraic methods can be used in arithmetic with the result of making processes appear more rational, thereby breaking up the mechanical method of presenting matter, and at the same time preparing for formal algebra those who are destined to take it. For example, pupils always seem to be interested in discovering a new way of doing the following example, $7\cdot 5 + 6\cdot 3 + 6\cdot 7 + 8\cdot 3 + 9\cdot 7 + 3\cdot 11$, that is, instead of finding the value of each term in order, and adding, taking them by 7's and 3's, $5\cdot 7 + 6\cdot 7 + 9\cdot 7 = 20\cdot 7 = 140$; and $6\cdot 3 + 8\cdot 3 + 11\cdot 3 = 25\cdot 3 = 75$, and $140 + 75 = 215$. This is an important algebraic principle which many authors even have overlooked, and pupils seem to like it; it can be used in addition and subtraction:

$$\begin{array}{r} 11\cdot 7 \\ 3\cdot 7 \\ 15\cdot 7 \\ \hline 29\cdot 7 \\ 203 \end{array} \qquad\qquad \begin{array}{r} 8\tfrac{1}{2}\cdot 13 \\ 3\ \ \cdot 13 \\ \\ \hline 5\tfrac{1}{2}\cdot 13 \\ 71\tfrac{1}{2} \end{array}$$

Few pupils coming from the grammar schools can add a long column in groups of 10 which is so much more rapid and which helps them in algebra; and the majority of them are quite surprised when told that the easiest way to subtract is to add, very much like cashiers do. This idea is very helpful both in arithmetic and algebra, for example,

$$\begin{array}{r} 15\tfrac{2}{3} \\ 6\tfrac{3}{4} \\ \hline 8\tfrac{11}{12} \end{array} \qquad\qquad \begin{array}{r} 8\tfrac{1}{2} \\ -\tfrac{3}{4} \\ \hline 7\tfrac{3}{4} \end{array}$$

In order to get $15\tfrac{2}{3}$ it is necessary to add to the subtrahend

$$8 + \frac{1}{4} + \frac{2}{3} = 8\frac{11}{12}.$$

In the second example it is necessary to add 7 and $\frac{1}{4}$ and $\frac{1}{2}$ to $\frac{3}{4}$ in order to get $8\frac{1}{2}$.

Multiplication can be easily explained and many examples made easier by the algebraic method. How many pupils in multiplying 47 by 5 would multiply the 40 first and then the 7, or how many really know that it is $40 + 7$ multiplied by 5, and it makes no difference which one is multiplied first; or that the quickest way to multiply 723 by 3 is to consider it as $700 + 20 + 3$ and multiply from the left. This same principle appears in denominate numbers, and this subject can be taught more rationally in this way, instead of by rule—I think that it is a mistake to ever introduce a rule.

$$\begin{array}{r} 5 \text{ yds.} + 2 \text{ ft.} + 8 \text{ in.} \\ 4 \\ \hline 20 \text{ yds.} + 8 \text{ ft.} + 32 \text{ in.} \end{array}$$

Each one must be multiplied, and then the pupil can change these denominations afterwards.

In multiplying 45 by 37 what we really do is to multiply the 40 and 5 first by 7 and then by 30 and add, but it can be done mentally very easily by beginning with 30, thus

$$\begin{array}{r} 40 + 5 \\ 30 + 7 \\ \hline 1200 \\ 150 \\ 280 \\ 35 \\ \hline 1665 \end{array}$$

To square a number by this principle is still easier as $(50 + 7)^2$.

Such a presentation of multiplication has three advantages, it shows the pupil a reason for the usual arrangement of numbers in multiplication, it gives him an easy way of doing examples mentally in less time than by writing, and it prepares him for the general principle of $(a + b)(x + y)$ if he goes on to algebra. This same principle is of great use in fractions, yet not one pupil in 75 can apply it to such an example as $12\frac{2}{3} \cdot 15\frac{3}{4}$, and I doubt if all teachers can, but why should not this method

be generally used? In the first place, how many pupils have had it called to their attention that $12\frac{2}{3}$ means $12 + \frac{2}{3}$? Mistakes are continually made in surds on account of this very thing; for example $\frac{2}{3}\sqrt{12} = 2\frac{2}{3}\sqrt{3}$. But to come back to $12\frac{2}{3} \cdot 15\frac{3}{4}$, which nearly all pupils would do by reducing to fractions, how easy a mental problem it becomes if taken as

$$\begin{array}{l} 12 + \frac{2}{3} \\ 15 + \frac{3}{4} \\ \hline \end{array}$$

The same is true of division, such as $12\frac{3}{4} \div 3$ or $27\frac{3}{4}$ by 4, and 125 yds. + 2 ft. + 11 in. divided by 3—there would be $\frac{1}{3}$ as many yds., $\frac{1}{3}$ as many ft., and $\frac{1}{3}$ as many inches, and then these could be changed afterwards. In an ordinary example like $2316 \div 24 = 96\frac{1}{2}$ (remainder 12) there is hardly a pupil who could not get the correct answer, yet hardly one who could tell *what* he really does with the remainder—he doesn't realize that he has been dividing the dividend by parts, and that the part remaining must also be divided by 24, giving $\frac{12}{24}$ or $\frac{1}{2}$, which must be *added* to the quotient as a part of it; from not knowing this, beginners in algebra get such quotients as $a - b\frac{b}{a+b}$.

The algebraic method of finding the H. C. F. and L. C. M. recommends itself on the same grounds as these other principles, that is, it makes the work easier, lends itself more readily to explanation, and is therefore more rational and less mechanical; $3 \cdot 5 \cdot 7 \cdot 2$ can more readily be divided by $2 \cdot 7$ than 210 can by 14, and they are the same thing in two different forms. So in the following example, the algebraic method can be used throughout

$$\frac{11}{21} + \frac{9}{10} + \frac{7}{15} + \frac{13}{14} =$$

$$\begin{array}{cccc} 3 \cdot 7 & 2 \cdot 5 & 3 \cdot 5 & 2 \cdot 7 \end{array}$$

$$\frac{11 \cdot 2 \cdot 5 + 9 \cdot 3 \cdot 7 + 7 \cdot 2 \cdot 7 + 13 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 5 \cdot 7}$$

$$= \frac{110 + 189 + 98 + 195}{210} = \text{etc.}$$

Fractions afford the greatest field for algebraic work. Cancellation as such should be abolished; pupils use it on all occasions, everywhere, and are usually wrong; cancellation seems to mean to them drawing lines through any two things that happen

to look alike. A great deal of such work can be prevented by keeping away from rules, and giving axioms as reasons for doing things. Often on asking a pupil what he has done and why he has done it, he points helplessly at his correct work and lets it speak for itself. For example instead of teaching, as is usual, that $\frac{2}{3}$ is changed to 12ths by dividing 12 by 3 and multiplying by 2, an axiom should be introduced to the effect that we can multiply the numerator and denominator of a fraction by the same number and not change its value, and then the only question is "by what is it necessary to multiply?" The same is true in reducing fractions to lower terms—if we make the parts 3 times as large we need only $\frac{1}{3}$ as many, etc. Pupils like to see this compared with adding the same number to numerator and denominator. When a pupil has seen these axioms he has a reason for doing things, and he also has something useful if he is going on to algebra proper.

In arithmetic there could be a more extended use of algebraic symbols, especially parentheses. Who knows what $30 \div 5 \times 2$ means unless some one has told him? If two successive examples could be written $(30 \div 5) \times 2$ and $30 \div (5 \times 2)$ the pupil would be made to think, and anything tending to this end is a "consummation devoutly to be wished"; also, is there any difference between $10 - 7 + 2$ and $10 - (7 + 2)$? Instead of the cumbersome sign of multiplication, the dot could be used as in algebra. Complex fractions, in which are involved so much patience and care, could be extended, and attention drawn to such combinations as $\frac{2/5}{3}$ and $\frac{2}{5/3}$, etc.

Arithmetic and geometric progressions could be introduced in an elementary way as an interesting side light on insurance and interest, as well as in other ways, and even very young children could get lots of pleasure out of combinations and permutations if adroitly handled—give them pieces of card-board in four different colors and let them see in how many different ways they could arrange the four, or how many different groups of two they could get; or let the boys figure out how many different base-ball batteries they could make up from their old arithmetic friends John, James, Henry, and Wm. as catchers, and Reginald, Percy, Fortescue, and Mike as pitchers; and after they have found out a few of these things, tell them that the seven boys in

the first row could be seated in 5,040 different ways, or that the captain of an eight-oared crew has a choice of about 40,000 different ways of arranging his men, or that the ten books on the teacher's desk might easily get out of order, as there are 3,600,000 different orders that they could be in. A grilling and gruelling recitation in algebra or arithmetic might very profitably be stopped by a gentle transition into a discussion of the various methods of signalling on a warship, or into a lesson on word-analysis which is so sadly lacking from all of our courses—how many teachers have time to call attention to the endings of addend, minuend, dividend, multiplicand, or show that a dividend is so called for the same reason as some of the things that the Standard Oil declares; and would it not be true that pupils would be less apt to confound subtrahend and minuend if they knew that the former came from "sub" and "traho, or tractum," and for the same reason, less prone to say that the product of 3 and 5 is 8 as some invariably do in every class that comes into the high school? If a pupil is asked to go down one certain stairway out of several, he is more likely to remember which one to take if he knows the reason for going that way. It is such side-stepping from the regular hum-drum work of school that gives the pupils some interest in their work, and broadens their view, and a teacher who can pause long enough in the mad rush for examinations to present some such alleviating information in an interesting talk is really a "superior teacher" and is doing the pupil more good than one who can force ten times the amount of *knowledge* into the pupil's head with the result of disgusting him with the subject, the teacher, and the school; *such* "superior" teachers, and there are many of them, should be sought out and besought to radiate their effulgence on their less "superior" comrades.

Just one more topic, and that is graphs. As used in elementary algebra they are simply a kindergarten method of representing only crudely and more or less clearly, usually less, something which is already as clear as it can be made; in algebra

$$2x + 3y = 5$$
$$3x + 2y = 7$$

means that there are in existence two numbers such that two times one of them plus three times the other equals five, and

three times the first plus two times the second equals seven; and then there is a clear way of finding what those numbers are. To invent an entirely new system in order to put a new interpretation upon what is already clear, seems unnecessary; yet it is interesting, and should by all means serve as a side issue to be followed up by an elementary discussion of curves; for any pupil in either the elementary or high school is interested in learning how to make an ellipse with a cord, and to find that it is really only a flattened circle with two centers instead of one, and that the sun shining through a round hole in the shade casts an ellipse on the wall and that the curtain cord forms a catenary, and that a piece of tape on the tire of a moving bicycle goes like this ⌒⌒⌒—and dozens of other things which make the pupils observing. But the real place for graphic tables and curves of all kinds is arithmetic; here is a great field for the tabulation of all sorts of interesting features and statistics—the boys could arrange a table for the base-ball league or interest tables for varying sums, time, and rates, tables of railroad fares, multiplication tables, etc. An ingenious teacher could make such work tremendously helpful and not boresome, and in connection only with a versatile teacher of English could give a boy a more useful education than he is getting now—more useful because it could be made more reasoning and resourceful.

CURTIS HIGH SCHOOL,
NEW YORK CITY.

SHOULD FORMAL GEOMETRY BE TAUGHT IN THE ELEMENTARY SCHOOLS? IF SO, TO WHAT EXTENT?

By D. J. Kelly.

In appearing before this assembly I feel somewhat like an impostor, for I am not a mathematics teacher nor have I ever been one. Neither do I make any claim as a mathematician but am merely a plain superintendent of schools, somewhat young in experience and a trifle old-fashioned in ideas. As such I speak this afternoon and should you disagree with anything that is said you are at liberty to do so for "my hat is not in the ring" nor have I any fears of "recall."

Since I was asked to speak as a superintendent of schools it seemed best for me not to give my individual opinions and experience alone but rather to stand as the representative of school superintendents as a class and to give you an insight into the general trend of the thought and practice of those who are administering the affairs of our leading schools at the present time. Consequently I prepared a questionnaire and submitted it to representative cities in every section of the United States—from Maine to California—from Maine to Louisiana. I received over a hundred replies which have been the material largely used in this discussion, hence, the voice that now speaks to you is not that of a single prophet crying in the wilderness but rather it is the blending of many voices—a sort of composite made up of the well-modulated tone of the conservative East, the lusty yell of the progressive West, the discordant note of the insurgent Middle West and the gentle murmur of the awakening South.

As is usual in such cases many and varied were the opinions expressed and the testimonies given. Some seemed to misinterpret the intent and purpose of the questionnaire, evidently believing that I had some ulterior purpose in view—possibly to inflict on the schools of this country some new form of persecution or inaugurate a new style of faddism. Some apparently thought that the committee who assigned me this topic did so

for the purpose of testing my sanity and that the school superintendents of this country had been designated as a committee to pass judgment thereon.

Now I disclaim any ulterior intent, for I never meant to suggest or inflict anything cruel, unreasonable or foolish but I was merely seeking the light for the purpose of passing it on to shine over or into this assembly. Furthermore, in considering my questions no superintendent had any right to pose as an alienist and offer expert opinion as to my mental condition. One informed me that the topic was too absurd for discussion and another frankly stated that he thought I must be crazy to take up such a topic. I protest that I am not crazy but rather am clothed in my right mind entirely and that this topic is a most sensible one even for school superintendents to consider.

Before giving you the result of my findings I wish to treat the subject briefly from the more technical or scientific point of view. Those of you, which are doubtless all of you, who have read Dr. Smith's most excellent book, "The Teaching of Elementary Mathematics" have found that the author gives a very illuminating treatment of this very subject which we are now considering and in this treatment we find a summarization of the opinions of various famous psychologists and mathematics teachers on the teaching of geometry in the elementary grades. From these I wish to quote briefly.

Rousseau held that "the elementary concepts of the science of geometry should be acquired in the lower grades but for these pupils it should come by the art of seeing instead of by the art of reasoning." La Croix a leading mathematics teacher of a hundred years ago said, "Of all branches of mathematics geometry is possibly the one which should be understood first. It is a subject well adapted to interest children provided it is presented to them with respect to its applications. The operations of drawing and of measuring cannot fail to be pleasant, leading them to the science of reasoning." Laisant believed "the first notions of geometry should be given a child along with the first notions of algebra following closely upon the beginning of theoretical arithmetic."

As Dr. Smith states these are not the ideas of mere theorists but of practical teachers and these ideas have been carried out with more or less extent in European and American schools.

In this country Professor Hanus worked a course for the seventh and eighth grades of the Cambridge and Boston schools. It began with object teaching and led up to the demonstration of all the simpler propositions. In Germany a course was prepared somewhat similar in scope but neither of these savored very much of Euclid. Some formal demonstrations, however, were introduced as a sort of climax.

In this country at least results could not have been satisfactory, for Boston reports that they now teach no geometry in their elementary schools nor do they believe it desirable. Superintendent Parlin, of Cambridge, says that formal demonstrations should not be taught in the grades, but he does advocate a sort of inspectional form which will lead pupils to discover by inspection much that they will later demonstrate by the usual methods.

I find that Galesburg, Illinois, also had, about forty years ago, a course in geometry in their grades, which was subsequently discontinued.

Nashua, N. H., tried it in the ninth grade about fifteen years ago, but being only partly successful it was dropped. The children's minds seemed too immature to grasp a subject of this kind.

Jersey City has given inventional geometry in the seventh and eighth grades, but dropped it as beyond the comprehension of the children.

The only place I found now having an elementary course in the subject is Indianapolis, which introduces it in the seventh grade. This course is almost wholly of a constructional and practical nature, without any formal demonstrations, unless one could call the answering of questions such.

Concord, N. H., has taught concrete forms from Hornbrook's Concrete Geometry with favorable results. In St. Louis geometric concepts are introduced through arithmetic text-books and in drawing. Auburn, Maine, hopes to include it later. The Des Moines superintendent states that he has had foreign children come into his schools who had had geometry in the grade schools of their native country, but their knowledge was so superficial as to lead one to believe that it is a pedagogical blunder.

On this point of the pupils' ability there seems to be a great difference of opinion.

Dr. Leonard, of New Rochelle, says that there is no question but that children can easily learn the simpler propositions in formal geometry but there is more useful work for them.

Superintendent Shear, of Poughkeepsie, believes that geometry may well be taught in the eighth grade. Children should know certain forms such as lines, angles and polygons. In addition there should be a few simple proofs such as: (1) vertical angles are equal, (2) the sum of the angles on one side of a straight line is equal to two right angles. It does seem to me, if I may venture an opinion, that children can comprehend the demonstration of either of these as easily as they can the computing the rate of income on 40 Chicago, Rock Island and Pacific refunding 4's purchased at 96¼ or the time in Honolulu when it is sunrise in New York.

In opposition to these views I find a rather formidable array. Yonkers claims that the precise methodical reasoning of formal geometry is too formal for young children. Other places that agree with this testimony are Ithaca, Oswego, Kingston, Albany, Atlanta and Denver.

Oakland, California, believes that below the high school geometry would be almost wholly a memorizing process. Superintendent Tisdale, of Watertown, N. Y., agrees with this and sounds a note of warning. He believes that in substituting memorizing for reasoning the children would get a false start and thus do much harm to their further progress in mathematical studies.

There seems to be a very prevalent opinion that formal geometry has no place in the elementary curriculum for different reasons. Some superintendent, as has already been stated, believes that it could be done, but he regards it as unfeasible. As Superintendent Smith, of Cortland, says, " A boy may be taught to walk on his hands, but there is no reason for it so long as he has good feet." Newton, Massachusetts, says that it might be taught but so might a host of other things.

Other subjects are suggested as giving just as good training besides being of more practical value. Training to think, to reason and to observe could be carried on as well or better in the study of English grammar, arithmetic, science or almost any other subject. In fact a most successful superintendent in one of the best cities educationally in this state claims that formal

geometry is of but little value anywhere in the educational process.

Many superintendents think it impossible because of lack of time. Rochester says that we are attempting too much in the grades and a vast chorus catches up the refrain and sing, "so say we all of us, so say we all." The elementary course is too crowded; our grades are overcrowded; there is no room for anything more; our schools are overloaded; no time for anything more; no more subjects in the elementary course; work should be more concentrated; we now spread out too much; let us concentrate; fewer subjects better taught; etc.; etc.

This may all be true, yet if we can offer something better than we now have should we not do it? We often do attempt too much and do much that is superficial. Leaving out the question of geometry we have to eliminate many things that might be attempted. Aristotle taught us something about the importance of exercising the sense of proportion. President Hyde, of Bowdoin, says that a proper development of this sense is a requisite for a well-balanced teacher. This does not mean though that we are to neglect a good thing just because we are busy. If best to do so take up the new thing and leave out something less important.

Another objection is one that has already been cited, namely, too difficult for the children. In addition to these the lack of suitable text-books and scarcity of properly trained teachers would make the task of introducing formal geometry into the grades impossible. It is true that text-books are wanting, but no teacher is fitted to teach any grade mathematics unless she has had a thorough training in geometry; without it no teacher can get the broad, comprehensive view of arithmetic needed to assign relative values and properly grade the work.

There seems to be a very general call for the constructional or inventional geometry, which one superintendent says is no geometry at all, but for the sake of argument we will call it such. That this is essential there seems to be little reason to doubt.

Even as low as the kindergarten some geometrical facts can be taught and in connection with arithmetic and drawing some geometric terms may be used. Our children must know the shape and names of figures and solids. The construction of squares, triangles and other problems is an interesting and

essential part of drawing. How to measure these figures and the solids is a necessary part of arithmetic. In the industrial work of the higher grades further application could be made.

It is my opinion that in connection with all of this informal work a properly trained and skillful teacher can easily bring in much that is usually regarded as formal with the result of giving the pupil more light, furnishing him valuable information which he may never get providing he does not enter the high school and if he does enter then his interest will be aroused so that high school mathematics will be looked forward to with pleasure and handled with greater ease. In the words of Superintendent Hughes, of this city, "if the teachers knew geometry so well that they could, out of the abundance of their knowledge work in some of the important principles it would be well."

For a general summing up I can do no better than to quote Superintendent Dyer, of Cincinnati: "Concrete or constructive geometry is a valuable part of the elementary mathematics and should be taught with mechanical drawing, wood work and mensuration. In the eighth grade a little formal geometry to give conception of the propositions in mensuration. Propositions of bisecting lines are easily within the comprehension of eighth-grade children. The formal should, however, be kept subsidiary, being included in the arithmetic and not given as a separate course in eighth-grade geometry."

And to the whole discussion I offer the following conclusions:

Informal or constructive geometry should be introduced into the elementary school work, (1) through the course in drawing in practically all grades, (2) through the work in arithmetic more especially in the upper grades, (3) through the manual training or other mechanical work, (4) through the knowledge, experience and skill of the teacher.

Formal geometry should not be introduced as a separate subject. It should be blended with the regular arithmetic of the eighth grade and given as a climax to the informal geometry and also as a prelude to the mathematics of the high school.

In attempting this we must have in mind the interests of the children, for their interests alone should be the end, aim and endeavor of all work in our schools.

ROME, N. Y.

AN ADDRESS.*

By E. S. Crawley.

Mr. President, Ladies and Gentlemen: I bring you cordial greetings from the parent Association of which you are forming a Section here, and from your older sister Section in the eastern part of the State. The President of the Association, my colleague, Dr. Schwatt, desired me especially to extend to you his congratulations and best wishes.

When Professor Long invited me to come here to speak at this meeting, he asked me to name my subject. I found it more difficult to do this than to think of what I wanted to say. In fact what I want to do is to speak of a number of things, all connected with mathematical teaching it is true, but not necessarily very closely connected with each other, or forming what could in any sense be called an address upon a set topic. What I shall have to say to you will not be novel. I can do no more than voice what all of you have thought at one time or another, because the experiences which we get from the pursuit of our profession must be pretty much the same everywhere, and our conclusions as to what is good and what is bad, its difficulties, its troubles of various sorts, and its pleasures and satisfactions, must be in the main alike. It is a good thing, nevertheless, for us to meet together and give utterance to our thoughts, even though we realize what I have just said, for it helps to give substantiality to our ideas. Moreover it is helpful to know that others are struggling with the same problems that vex us.

The tendency of teachers to form associations is not a very recent tendency, although some of us who have been in the harness for twenty-five years or more can recall the time when there were no such associations, or when their influence and activity were much more restricted than they have since come to be. The principal object of such associations is the promotion of efficiency. Apart from this they would have little reason for existence. That they accomplish this object is shown

* Delivered at the opening of the Pittsburgh Section.

by their continued growth, activity and ever-widening influence. I mention one thing in this connection, as it is a most unfortunate condition, but one which happily cannot well continue under the influence of an active and vigorous association such as this: I refer to the fact that there has often been in the past too much lack of sympathy between college teachers and school teachers. It has come about doubtless from the unfortunate relation they hold to each other in that the college teacher receives the product of the school teacher's effort and does not always find it to his liking. Being human he naturally says, or thinks if he doesn't say, unpleasant things about the teacher in the school. Doubtless this censure is sometimes deserved, and doubtless also if the product of the college teacher were judged in the same way by some one higher up, he would likewise often be exposed to well-deserved censure. You have heard the old joke on our friends, the physicians, that their mistakes are all buried, so in a sense is it with the mistakes of the college teacher: they are swallowed up in the hurly-burly of life and it is hard to trace them back to their source. Now, as a matter of fact, the teacher in the school has the harder, and, in some ways the more responsible task, and his failures should be judged very charitably by those who follow after. It is the school teacher's duty to sow the seed, to tend the young shoot, and to start it in the right direction. I wish I had words in which to tell you how strongly I feel the deep and vital importance of these beginnings in the study of mathematics. So much, so very much, depends upon getting the student to have the right point of view. Note that I do not say "giving" the student the right point of view. You cannot give it to him; you can help him to get it, but he *must get it for himself*. Of course, there is the technique, of which I shall speak later, but with the right point of view, or the right orientation with respect to the subject, if you prefer that way of putting it, there will be comparatively little difficulty, in most cases, with the technique.

Now a word as to what I mean by getting the right point of view. I mean simply that the student should be led to realize that mathematics is after all nothing but common sense. I am sure that time and again you have all been astounded at the evident helplessness of students, apparently in the full possession of their normal faculties, who have halted and stumbled

over the simplest deductions. I believe that this is almost invariably due to their utter failure to realize the common sense of the thing. Now as I have said, it is not enough to tell the student this, however impressively you may do it. He must get to see it for himself. That point attained, however, he is prepared to appreciate the beauty of mathematics, to enjoy its simplicity, and to see how each part helps the other and does its share in rearing the whole structure. In this connection the opening sentences of Dr. A. N. Whitehead's recent book, "Introduction to Mathematics," seem most appropriate and, with your permission, I shall read them. Dr. Whitehead says:

"The study of mathematics is apt to commence in disappointment. The important applications of the science, the theoretical interest of its ideas, and the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest. We are told that by its aid the stars are weighed and the billions of molecules in a drop of water are counted. Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it—''Tis here, 'tis there, 'tis gone'—and what we do see does not suggest the same excuse for illusiveness as sufficed for the ghost, that it is too noble for our gross methods. 'A show of violence,' if ever excusable, may surely be 'offered' to the trivial results which occupy the pages of some elementary mathematical treatises.

"The reason for this failure of the science to live up to its reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception. Without a doubt, technical facility is a first requisite for valuable mental activity: we shall fail to appreciate the rhythm of Milton, or the passion of Shelley, so long as we find it necessary to spell the words and are not quite certain of the forms of the individual letters. In this sense there is no royal road to learning. But it is equally an error to confine attention to technical processes, excluding consideration of general ideas. Here lies the road to pedantry."

The next question which naturally presents itself is whether

our teachers of mathematics are generally prepared to do this kind of work. I have no doubt that many are doing so; others perhaps need but a hint to set them thinking about it. There are doubtless many others, however, who are content to jog along teaching what the book says, and seeing that the pupils get the right answers without doing anything to vivify the subject or to articulate it with the facts of everyday life, as the current phrase goes. Now I, personally, am not so much concerned with "articulating it with the facts of everyday life"—that will take care of itself—as I am with articulating it with the laws of everyday common sense, by which, to explain further, I mean leading the student to make his mathematical thinking a spontaneous part of his mental activity, just as his thinking about football, the purchase of a new hat, or any other matter which does not require conscious mental effort. Many teachers, however, are helpless in an undertaking of this kind, for they, themselves, lack the necessary foundation, but I am aware that they are often the victims, perhaps the unwilling victims of a bad system. Let us suppose a case: Suppose in a school there is need, in an emergency, for a teacher to take charge of a class in Latin. The head of the school in looking around for someone to fill the gap finds a teacher who has vacant time at the periods when the class in question is to meet and asks him to step in. He demurs on the ground that he has studied very little Latin, that he has never read the author with whom the class is engaged, and that besides this, it has been so many years since he has opened a Latin book that he forgets almost all he ever knew about it. Can you conceive the head of the school insisting that notwithstanding these handicaps the teacher take the class? I will not say that this could not happen, for it is said that all things are possible, and I suppose interlinears are still in the market, but I think you will agree with me that it is very unlikely to happen. Yet I venture to say that when the subject is algebra or geometry, even more, if it be arithmetic, the exact parallel does happen with painful frequency. Yet I maintain that in no subject are specially trained teachers more necessary for successful results than in mathematics. If our associations can open the eyes of principals and school boards to this fact, they will be doing splendid work. To teach from the book, and simply from the book, means inevitable failure in mathematical

work. No text-book is entirely satisfactory to the capable teacher. He will sometimes wish to amplify the subject matter, or to present it from a different point of view, and the mere fact of his doing so acts as a stimulus to his students, for they see that the subject is not a mere lot of cut and dried formulæ or theorems between the covers of a book, but a living thing about which one can think, and which one can handle for himself. Again, the capable teacher will never accept as a reason why a thing is so the statement that that is what the book says, or that is what the teacher himself has said. Dogmatic assertions, no matter what the authority, must be banished absolutely from the mathematical class room. This does not mean of course that we may not give our students information about facts the demonstration of which is at the time beyond their reach. Such information is in fact stimulating to an ambitious student as it points the way for him to the higher levels. There is nothing so discouraging because there is nothing so hopeless, as to have a student tell you that he has "finished algebra" or "finished geometry."

Just now I said that when the student has gotten the right point of view of the subject there will not be much difficulty with the technique. Of course it will not be all plain sailing. All that I can do is to point out certain matters which my experience as a teacher have brought to my notice, which lead me often to wish that the earlier training of some of the students I get had been conducted in some way differently. My theorizing as to what might be done may not be of much value, but it may at least form a basis for discussion. Speaking of algebra, the fundamental difficulty with the inefficient student seems often to be failure to comprehend algebraic notation. Algebraic notation so far as it goes constitutes a language, and I often wonder if enough stress is laid upon this phase of the matter. Every algebraic statement (*i. e.*, every algebraic equation or identity) and every algebraic expression can be translated into English, and every statement in words which has to do with suitable matters can be translated into the language of algebra. The former process is usually the easier, at least until a very complete mastery of algebraic technique has been acquired, so that in this respect the relation of algebra to English is the same as in the case of other languages. Now until a student can

readily translate any algebraic expression into its equivalent English, or better until he can without such actual translation form from the expression itself a clear mental picture of the meaning, he cannot be said to have gained a satisfactory mastery of algebraic technique. No student who looks at the expression $\frac{1 + 2x}{3 + 2x}$ with any appreciation of its meaning will say that we can cancel the 2's and get $\frac{1 + x}{3 + x}$, and yet there is no more common error with the indifferent, or the ill-prepared student. I wish by the way that the term "cancel" could be eliminated entirely from the vocabulary so far as it applies to the operation of dividing the numerator and denominator of a fraction by the same factor.

While in its expression algebra is appropriately denominated a language, in its operation it is a machine. Herein lies its power. In applying algebra to the solution of a problem we substitute its purely mechanical and easily applied operations for what would require otherwise a complex mental process. Students often, or perhaps generally, fail to appreciate this or only half appreciate it. When I was a boy in school we had once a week an exercise called mental arithmetic, in which we were called upon to solve without so much as a stroke with pencil or chalk a variety of problems by a process of pure reasoning. These problems were usually of a kind that could be solved much more easily, by which I mean with much less conscious mental effort, by algebra; and perhaps the recognition of this fact helped us to appreciate the value of algebra. I understand that such exercises are banished from the curriculum now, but I sometimes wonder if they might not be restored with advantage. Algebra, being a machine, must be correctly used. If one pulls the wrong lever, or turns a crank the wrong way, he will probably get some result, but not the one he should get, and if he is not aware that he has misapplied the machine he will be left in ignorance regarding the reliability of his result. The correct application of the machine requires careful observation and attention, and a habit of precision, all qualities not usually innate, which therefore must be developed. I have now in one of my classes a student who never by any chance seems able to see a minus sign if it is attached to the first term of an

expression. Another fault of the same general character which affects multitudes of students is what I term blindness of the left eye. This is an affection not ordinarily manifest, but breaks out, so to speak, only when the student starts to work transforming an equation. He will go through all sorts of evolutions with the right hand side of the equation, changing its value in all sorts of ways, apparently quite blind to the fact that the equation has a left hand side at all. This is especially true when the left side of the equation is a single expression, dy/dx perhaps, whose value he is trying to reduce to a simpler form. Another most desirable quality in the student of algebra, and in fact of any branch of mathematics, is a sense of order. When I see how some students do their work my surprise is not that they sometimes get things wrong, but rather that they ever get them right, so hopelessly lacking in any orderly arrangement is what they produce. One of my colleagues at the University of Pennsylvania has a very forcible way of bringing home to his classes the advantages of order in one's work. He rapidly puts on the black-board twenty-five or thirty crosses arranged in no special order, but hit or miss all jumbled up together, and asks if any of the class can tell at a glance how many there are. Of course no one can. He then arranges the same number of crosses in rectangular order, and the number is at once disclosed. He tells me that this simple expedient has been very successful in impressing the students with the value of order.

Students quite as a matter of course fail to appreciate to anything like its full extent the power of the tool called algebra. I say, quite as a matter of course, because it is natural that this should be so. We as teachers usually expect too much. We forget the enormous advantage which we have through our familiarity with the processes due to the constant repetition of them which we are obliged to make. Until such a facility in handling algebraic expressions has been acquired as to make the operations a matter of second nature, so to speak, one cannot give his entire attention to the real meaning of what he is doing. And in this connection, I believe that the element of time plays a great part. Most of you I think will be able with me to recall experiences where after working unsuccessfully at some problem or discussion without success, and leaving it in disgust, you have come back to it after a week or a month, or

perhaps longer and find to your astonishment that there now appears to be no difficulty at all. Your mind has in the meanwhile been unconsciously adjusting itself to the situation and the correct point of view has been reached. So with students time is needed to assimilate the new ideas which the study presents to them, and if it is not given, mental indigestion follows. For this reason I think that in the study of a subject like algebra as much can be accomplished with two periods a week for two years as with five periods a week for one year.

I find amongst the students who come to me two faults, or rather one fault and one error of judgment prominently in evidence. The first is inability to separate what is essential from what is merely incidental in an operation. Very briefly and simply what I mean is this, and I shall use numbers to illustrate my point for they will serve quite well. If a student is asked to multiply 20 by 18 and divide the result by 45 times 8, he often does not realize as fully as he should that when he has written $\frac{20 \times 18}{45 \times 8}$ the operation is actually completed, and that anything he may do subsequently is merely a transformation of the result in a simpler form, that is, it is merely incidental. The second point to which I refer above is a tendency to do a great deal of unnecessary multiplying of algebraic expressions. In most of the operations of algebra, it is better to leave an expression as the expressed product of factors, for then one can see how it is made up, and I constantly find it necessary to warn students not only against the futility, but of the positive harm often done by needlessly performing expressed multiplications.

I find my students as a rule lamentably ignorant of the meaning of the technical language of algebra. Such words as "term," "rational" and "irrational," "imaginary," "root of an equation," "degree of an expression," and various others very often convey no meaning to the student's mind, or at any rate no clearcut definite meaning, so that to make use of such expressions in talking to a class is often sheer waste of time. Another point to mention is a difficulty arising out of the fact that as we advance in the study of mathematics our way of looking at things shades over very gradually from one point of view to another, and the student does not always follow as he should. This statement is very vague and I will try to be more specific.

In the study of elementary algebra, for example, x represents to the student some unknown number, with the emphasis upon the adjective "unknown." But as we advance we change gradually our point of view and begin to think of x as a variable, until finally we learn to think of it exclusively as a variable and the question of whether it is known or unknown is often if not usually of quite minor importance. The introduction of the graph into algebra is an easy and effective way to make this transition. But although graphical methods have now been in vogue in elementary teaching for some time I still find that while students can generally solve a quadratic equation, they are not well acquainted with it as a rational quadratic function of a variable which vanishes for two values of the variable, called the roots of the equation, nor do they readily handle this quadratic expression in the different ways in which it is used, ways which have no connection with its actual solution.

Mathematicians have in recent years been profoundly interested in the study of the fundamentals of both algebra and geometry. While as teachers it is our business to know something of the results of these investigations on account of their great significance and the enlarged view they give us of mathematical operations, we cannot introduce them except with great caution to our younger class of students, and in this class I would include the generality of students in college as well as those in school. They are rather for the selected few. This leads me to make the remark that I deem it of doubtful advantage to bring to a student's attention matters which either through his immaturity of mind or want of preliminary training he is not yet ready to appreciate. It can serve only to befog him and to raise in his mind an impression that, however it may be for others, he has reached the limit of his powers of comprehension of so abstruse a subject. Similarly it is useless to expect students to appreciate refinements of demonstration intended to give scientific completeness, when the special difficulties which such refinements are intended to meet have not occurred to the student himself, or when if pointed out to him he does not grasp their force or significance. On the other hand there is danger in holding off too much from such matters, as reference to them sometimes plants the seed of strong future growth. The wise course to pursue requires judgment and

tact. The teacher of elementary algebra encounters a situation of this kind when he takes up imaginaries. To a great many students $\sqrt{-1}$ is bound to remain always merely an "imaginary" or impossible number, but to some it can be revealed in its true significance as the door leading to a new and fruitful field of endeavor, for it leads to a new algebra, a more extended number system, where our old unit of everyday life is not the only unit. Thus the ideas are broadened and made receptive to still wider generalizations to come later.

No more striking results in mathematics have been attained than those which have followed from investigations into the foundations upon which geometry rests. The geometry of the school, however, must remain, as it has been, the geometry of our experience. As a result of these investigations, however, we are more ready than was the past generation of teachers to admit that experience and intuition play a significant part in the study of geometry. We do not insist so much as formerly upon definitions of such concepts as point, right line, and angle, because we realize that the pupil knows by intuition and from experience what they mean. In fact no satisfactory definitions of these and similar concepts have as yet been given, and perhaps never will be given. So also with many propositions. In short, we recognize, or perhaps I would better say we frankly acknowledge, that geometry is not such a perfect bit of logical deduction as we have been inclined in the past to consider it. I would not have you think that I am disparaging the value of geometrical reasoning; far from it; I am only trying to point out that we are willing to readjust our views as to what it is permissible to accept as the foundation upon which to build up our logically derived series of theorems. The difference is that we now know that as we vary the foundation we vary in many respects the superstructure, and that we actually have a choice in the matter of the foundation. It remains true, however, that for the geometry of our high schools the foundation chosen must be that of everyday experience. If you have not read the Provisional Report of the National Committee of Fifteen on Geometry Syllabus, I would advise you to do so. The historical introduction is especially important and suggestive. The committee in their suggestions as to details, that is, in the list of propositions and their classification into those requir-

ing rigorous proof, and those for which only informal proof is desirable, lean away from the side of rigor rather more than seems to me personally to be wise, but this is a question which each teacher must decide for himself. My feeling is based not so much upon consideration of the requirements of the subject as it is upon my experience with students of careless or inexact mental habit. So many students show these qualities in their mathematical work that I shrink from anything which may encourage it. Insistence upon clearly stated geometric demonstrations is a foe to inexactness.

I often wonder whether we do enough in our teaching of mathematics to impress upon our students the surpassing beauty and profound reach of the science. I ought not to say that I wonder whether we do this, because I am well assured that we do not. It is not unusual to hear the pursuit of mathematical study decried as being dry, uninteresting and dehumanizing. It is true that the necessities of the case require that we devote most of our time and effort to mere drill in the mechanical operations, and it is not to be wondered at that students become lost in attention to detail, and fail to get a broad, general view of what it is all about. The teacher standing above, as though on a hill top, can see the distant view, but the student grubbing in what to him is more or less of a jungle, if not a jumble, does not get this view. The teacher should try now and then to give him a glimpse ahead so that he can, to some extent at least, see things in their true perspective. One of the old Greek mathematicians is reported to have said, "The Deity continually geometrizes." This, I take it, was only another way of stating that the whole physical universe "lives and moves and has its being" on a basis of mathematical law. This defines the scope of our science, and if we endeavor to bring our students to an acquaintance with this which is the real incentive to mathematical study and investigation, we shall, I think, be rewarded with a more general interest and with an occasional burst of enthusiasm. I remember reading in a little book describing life at Yale at about the middle of the last century, a little story which has some point in this connection. It was the custom at that time for the different tutors in the College to take turns in leading the services at morning chapel, and upon one occasion, when one of the mathematical tutors was officiating in this

capacity, he began his prayer in the following words, "O Thou who guidest the motions of the heavenly bodies as they move in their appointed orbits, the force varying directly as the mass and inversely as the square of the distance," . . . Poor fellow! It was ludicrous enough we must admit but at least we can see that he did not lack an appreciation of the dignity of mathematical principles and their far-reaching importance, although we should probably prefer to use other opportunities of bringing these things to our hearers' attention.

University of Pennsylvania,
Philadelphia, Pa.

WHAT SHOULD COMPRISE THE SUBJECT MATTER OF THE ARITHMETIC IN THE ELEMENTARY SCHOOLS AND IN WHAT GRADES SHOULD FORMAL ARITHMETIC BE TAUGHT?

By H. J. Wightman.

The Lord, the school and society are responsible for the type of individuals that gets into the high schools, and after the Lord and society have done all that we can expect them to do for some time to come, there is left a much larger problem than simply to find the G.C.D. or the L.C.M. The child is an active thinking individual, if we do not suppress his activity and mechanize his thinking and convert him into a jumping-jack which responds only as the teacher pulls the strings and then apparently in a way that suggests need of lubrication. I have nothing but pity for the child who is allowed to think only through the ruts made by the juggernaut of mechanical teaching. Formal mental discipline, as interpreted by the Gradgrind martinet with its memoriter and rule-stuffing accompaniment, has been the fetish which has blocked the road for the development of childhood in mathematics for a long time.

It is no wonder that teachers of high schools say that the pupils they get are unable to take the initiative in any new type of reasoning and are unable to generalize and get the essence out of any considerable body of data. And these high schools get only the cream of the *academic-minded* pupils, whereas the grades are dealing with all types of pupils during the compulsory attendance period. The human weeds found in the grades are apt to interfere with the development of the type destined to reach the highest intellectual fruitage unless wise provision is made for their early segregation.

In this paper I shall emphasize aim and method rather than matter, because through aim and method, which will determine the matter and which will recognize the type of individual being dealt with, must come our salvation and our future better practice and results.

Elementary school arithmetic work divides into two general divisions in conformity to the psychological or natural division of the grades from the basis of a child's development, into the primary four years, and the grammar four years.

Formal work, as I interpret it in the topic given me for this paper, really begins with the grammar grades, the second four years. Formal work to me means logical step reasoning on problems (not process work or example solution). I would thus make formal work begin at the time that pre-solution statements begin to be emphasized. Mechanical solutions may be formal as to the machinery of production of answers, but they involve no reasoning on the part of the pupil, or so little consecutive logical thought that they are purely process imitations. The pupil acquires process forms through imitative doing from the very first, but reasoning is not harnessed to the performance until years later. We must always be mindful of the fact that one does not gain strength or power through the activity or exertions of another. We must also recognize the fact that children cannot assimilate the conclusions or rules worked out by any text-book-maker simply through memorizing the same and using them in a routine mechanical way to solve problems. We must not allow teachers or text-books to do all the organized thinking for the pupils and to present them with conclusions or rules to pigeon-hole in their memories. A pupil's reasoning must act upon the raw material of problems to make the why of doing clear. A rule, as well as a definition, is an abomination where it is the beginning rather than the conclusion of pupils' personal work and effort. To make myself clear perhaps this last statement needs explanation. Formulas like $2\pi R$ and πR^2, borrowing and carrying in subtraction and addition, inverting the fractional divisor to simplify work by multiplication, cancellation, etc., are simply taken as process data by the child, and reasons for their use are reserved until the pupil is ready to understand without a serious waste of time. A child of the fourth grade can gain skill and fix the operation or process of dividing one fraction by another, but it is unpedagogical to attempt to force the why upon him until one or two years later. To all of us some things clear up long after we have passed the performance stage. The child may learn a process through doing long before he is able to understand the whys for every

step, but his mental machinery should not be clogged by the meaningless verbiage of memorized rules.

The *primary work* must have as its basic trend conformity to the natural or physiological growth or development of the child, so that it will make the most of the keen memory of this period in fixing data and tables which must be rooted deep and firmly, and so that it will not attempt to force logical reasoning at a time when it will but blunt and stunt development.

The *grammar work* must be responsive and responsible to the every-day practical life demands, adjusted, to some extent, to fit the community experiences of the child.

In *primary work* the child occupies the center of the stage and all lines of effort converge to him rather than to the subject.

In *grammar grade work* the subject shares the arena and the method takes on large importance. As the express train is superior to the old Conestoga coach for transportation, so is one method, chiefly because of the aim back of it, superior to another in training children to initiative thinking.

The *primary work* must make real and clear the fundamental number relations chiefly with the play motive as the medium.

The *grammar work* must perfect skill in fundamental operations and establish processes for practical solution of all needed forms of problems. In this the preparation for high school is only a by-product of the preparation for life, not the aim of grade work.

Both *primary and grammar grades* give children power through much practice in getting their own data for problems, and in making problems, in order that they may see the conditions in their right relations in individual problems when taken from books.

Both *primary and grammar* grades give much practice in doing problems in distinction from simply working problems, in order to properly motivate the work, make it real, and make it function in commonsense results. For example, we want problems such as: Find the number of gallons of water that this tank or that tank will hold. Find the number of pupils' desks that can be legally placed in this room under the requirement of 200 cu. ft. of air per pupil. Get the proceeds of this note or of that note which will be put into the hands of pupils without further comment. Find the cost of reflooring this

room, or any other room specified, with first quality white maple, purchased of a definite dealer. This doing of problems in which pupils must take the initiative in ascertaining data as they would have to do in real life brings a touch of reality and interest into the arithmetic work that is absolutely vital and which adds tone and quality to the thinking and to the results. All right reasoning must come from a comprehensive view of the facts involved. We cannot always confine pupils to problems touching their limited experiences, but we can bring with the new problems new experiences if pupils find their own data. We cannot educate children against their will, and they have little will to acquire and little desire to work until they have fixed a definite motive as a result of interest. We must not, however, fail to note that problems which may be real and concrete to adults may be very unreal, foreign and out of time for pupils of a particular grade. Please mark that doing problems is a very different thing from working problems and develops an entirely different fiber of reasoning; the doing of problems develops power for initiative thought because it demands initiative thinking.

In the *primary grades* the forms of practical expression and solution including a knowledge of signs and terms are fixed. Solutions should be in direct, short-cut, business forms. It is foolish to express a series of numbers with plus signs between them when these numbers must be rewritten in columns for solution. All algorisms should be practical and economical of effort, at least in their habitual form. There is a field for someone to work out uniform algorisms that shall satisfy first psychological, and, second, service needs.

In the *grammar grades* the complete statement or expression in equation form of the entire problem before any work is done requires the pupil to think through the problem and note conditions fully, before there is any juggling with figures. This complete statement of most problems develops a more independent, rational type of thought power and very materially reduces, through cancellation, etc., the actual work necessary. The complete statement of problems as a method is vital also in that it overcomes the thinking by driblets which requires six dozen questions from the teacher to unloosen a conclusion that should spring from one or two definite questions, provided that

the pupil has sized up all the facts of the problem in their relations to each other and particularly to what is required. The complete statement trains pupils to turn their search lights on the conditions of a problem as an individual reality and think at it until they think through it and rivet a conclusion. It prevents the thinking in grooves as the adding machine works. Instead of producing pattern thinking or doing as working from rules and parrot explanations does, the complete statement produces constructive thinking which means independence in handling mathematical data. It is one means to get the pupils to think on their own hook and to think determinedly and not by starts and spasms.

The complete statement of arithmetical problems is the one most important factor in arithmetic for preparing pupils for algebra and geometry work. It overcomes the most serious cause of mistakes or errors by demanding an understanding of the conditions and language of a problem before trying a hit or miss solution. It frustrates any desire of teachers to waste time and distract attention of pupils by written step explanations during the time of solving.

In both *primary and grammar grades* the problems and work must enlist the pupil's real wish to do in the same sense that a game of ball gives him motive for self-activity. The character of the problems worked, particularly in the primary grades, needs radical change to meet the psychological rather than the logical or formal standard, otherwise it lacks the proper appeal to the child.

In both *primary and grammar grades* the natural element of competition comes in to help secure accuracy and speed. There needs to be exercises with time limits and time records as in a 100-yard dash. We need in teaching to make use of those forces which naturally impel pupils to self-exertion in their free hours; we need frequently intensity of mental attitude and attention; we need more active and less passive attention.

In both *primary and grammar grades* there should be close correlation with recreative and constructive exercises.

In both *primary and grammar grades* the home work should comprise exercises for skill and drill, but never new work not thoroughly understood. Home work should also aim to eliminate individual deficiencies in order to bring pupils into right

working relations with their fellows. It is archaic to expect every pupil of a class to do the same home work irrespective of their personal needs. Explanations, as well as attention, need to be focused upon difficulties.

If my hands were not tied, largely by tradition, I would put into operation a course in arithmetic, enlarging upon the fundamental idea that can be observed to-day in the city of Gary, Indiana, in which the data of arithmetic, the terms and tools of solution, and the processes of operation are acquired through the self-activity of actually doing real problems. As an example; I would require the eighth grade or finishing class to go into the town, select a house that they would wish to build, take their own measurements; stake out foundations on the school lot; compute the cost of building cellar and walls of house; get actual prices on materials; draw their own rough scale plans; floor and roof, plaster and paper the house; construct the chimneys, fire places, etc., etc. All of these things pupils would do from their own measurements and data. They would ascertain from material men the actual cost of materials and eventually would arrive at a definite cost for construction of a house of the type selected and would compare this cost with the actual cost of erection, checking their errors in estimates. I would have pupils borrow money on notes and through building loans; take out a mortgage on the property; have it insured; make out tax bills, etc., and through these operations get a review of all the phases and operations of arithmetic. Through such real work the development of the processes and the opening up of the subject comes by actually doing, which is the vital point in arithmetic preparation.

In manual training we have two types of teaching. In one we see pupils placed in rows on benches with notebooks and pencils and the teacher standing before them lecturing and dictating in reference to the parts and uses of chisels, planes, saws, hammers., etc., giving them definitions and rules of construction. In the other type of school we see the pupils determining upon something practical to make and learning all they need to know about the use and care of chisels, planes, saws, hammers, etc., through actually using these tools under the watchful eye and direction of the teacher. These pupils in the latter type of

school are doing something which has motivated the child's energy and his interest.

In arithmetic we see the same two types of teaching. In one, a class is riveted to recitation benches reciting on rules and definitions, working book problems that rarely hitch to the child's interest, getting rather mechanical, formal and indifferent results. In the other type of school we see pupils turning their thought power upon the actual conditions of real problems which they wish to solve, going into the fields, into the workshop, into the town, or wherever it may be, to get their data in the natural way, coming together with enthusiasm and interest in accomplishing something real, and eventually arriving at operations and methods of procedure and such definitions as may seem to be needed. I need not comment upon which type of work gives the highest type of result as to tone and quality.

My experience in Altoona, Pa., in connection with a group of retarded or slow-developing grammar grade pupils that were given an opportunity to go into the school machine shops and take a course in practical shop mathematics, leads me to feel that if we get our pupils away from book work a great deal more than we do and bring them to real problems where they must do things and find some of their own data that we will accomplish an entirely different type and quality of thinking and will develop some originality and initiative in our pupils and will overcome most of the criticisms that are being heaped upon grade schools and will save to our high schools and higher institutions many that now become discouraged and fall by the wayside.

But as I am hampered by tradition and by a board of education who are hampered by tradition and customs of the past and as school men in general are so hampered, it is necessary for us to lay out a course in mathematics with general boundary lines and fences that do not shatter too rudely the ideas of our forefathers. We must, however, put in entry wedges as frequently as possible for the new type of work and particularly psychological methods and aims and with these thoughts I submit virtually the outline of work that I have arrived at after a lifetime of experience as superintendent and study of school needs.

First Grade.

The first grade work aims to give the child a clear and real idea of the number relations in the way that he naturally acquires facts, through pleasurable activities, games, relaxation and occupation exercises. These number relations are confined to the number as a whole and to the plus, minus, times, division and fractional relationships up to and including 10.

Baby talk is tabooed. Whenever a term is needed the term, as it will be used later, is given; thus the child from the first uses plus, minus and all the signs of operation without conscious drill on these terms. All terms of method, process and operation are used incidentally as a wise parent would use the word chair in implanting in the baby's mind the idea. He would say: Sit in the chair. Stand by the chair. Put the book in the chair, etc. The child learns the word chair through its relationships with his life activities. Nothing is more deadening and dwarfing to the thinking power of the first or second grade child than mechanical, formal arithmetic drill work.

To illustrate what I mean by games to establish number relations I give one of a great variety of exercises used. Some child is chosen as a leader to play wild horse; he runs down the aisle and taps five children who come to the front of the room prancing up and down. The leader announces that he captured five wild horses which he put in a pen, but three of them have jumped the fence and ran away. The teacher places on the blackboard $5 - 3$, and when the pupil completes his statement by saying that he has 2 horses left, she puts down the 2 after the equality sign on the board and reads the statement 5 less 3 are 2. Later on the pupil will write the whole expression himself which merely represents what he has done just as the sentence, John hit the ball, represents something that was done right in the school room before the teacher wrote the sentence on the blackboard. The teacher asks the pupils to listen and repeat what she does and then she raps 5 times and then says minus and then raps three times. The pupils do the same, then they run to the blackboard and write $5 - 3 = 2$. The purpose of this is to fix the idea of subtraction, not as a bookish fact, but as a reality and as something that the child uses and can use. The data and figures that a child must later have are

simply thrown in as incidentals and the child comes to know that these forms are used just as words are used to express certain relations of things that he deals with. The resourceful teacher will bring in many things that are of vital interest to her particular children and will teach the addition relation, the multiplication relation, the subtraction relation, the division relation, the partition or fractional relation entirely through the games and exercises that will furnish activity, physical relaxation and abundant pleasure and enjoyment for her pupils.

I have no use for the "Nancy" exercises in arithmetic, such as abstract 2 and 1, 2 and 2, 2 and 3, 2 and 4 in rotation around a class during which pupils are glued to their seats. I want some virility to the work from the very first minute. Instead of making school life hateful, arithmetic is one of the very best subjects to give the pupils the freedom of the class room and school and take away that timidity and restraint that is such a barrier between home life and the first year of school life.

I have still less use for those dry bones of arithmetic so common in classes presided over by the formal, methodical drill master; I want some flesh and blood, some life and activity; I want some emphasis placed upon the psychological and natural treatment of the subject rather than upon the logical treatment considered from the adult standpoint. The first grade is the most important grade in the school system.

Second Grade. (Method most Important.)

Game work of the first grade continued.

Picturing and expressing on blackboard combinations that form numbers to 10 as occupation work. (Domino work.)

Counting by 2's, 3's, 4's, 5's, 6's.

Master times tables, including 6's.

Dividing by 2 up to numbers of 6 times table.

Counting by 2's, 3's, 4's, starting with 1, 2, 3, 4, 5, and 6.

Fix addition facts by practice in writing and adding three numbers of 3 or 4 figures each.

Subtraction should be treated as a correlative process of addition, so that when adding $3+5=8$ pupils will see $8-5=3$. (Subtracting by adding eliminates one set of tables.)

$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ of numbers to 10 which give integers as answers.

Trade questions involving change up to 50 cents as subtraction drill.

Inch, $\frac{1}{2}$ ft., 1 ft. and use of ruler in paper construction work and in measuring objects, and blackboard work in drawing to measure.

Multiplying, and testing results by addition.

Roman numbers to 20.

Fix habit of checking and testing answers.

Sufficient doing problems to make real and clear the data to be memorized.

Pupils make problems from abstract statements such as $4 + 3 =$; also problems based upon games or imaginary conditions. (Conversion of examples into problems.)

THIRD GRADE.

Continue occasionally game work so that children will not come to look upon number relations as a bookish matter.

Make automatic all the times tables including the 10's as well as all addition and (subtraction) and (division) facts.

Dry, liquid and linear measures with simple changes of unit of measure. Practical problems.

Reading and writing Hindu numbers. Pointing off as write. Roman numerals to the extent found in readers.

Addition of mixed numbers as

$$\begin{array}{r} 8\frac{2}{3} \\ +6\frac{1}{3} \\ \hline \end{array} \qquad \begin{array}{r} 9\frac{1}{3} \\ +14\frac{2}{3} \\ \hline \end{array}$$

Multiplication by easy mixed numbers.

Short division with divisor 10 or less. Picture problems, considering division as correlative process of multiplication.

Speed contests in multiplication and addition examples.

Practical review of and drill in all fundamental operations.

Emphasize comparison of values. Whole numbers and fractions with problems. Compare 2 with 4, compare 4 with 2, compare $\frac{1}{2}$ with 2, etc.

Time reading, calendar construction, table.

Practical surfaces constructed and computed—doing problems through construction work. Candy and glove boxes, mats, tiles.

From market reports find cost of articles used in home.

Bills, and work with money and decimal point in all the operations.

FOURTH GRADE.

This grade masters thoroughly the four fundamental operations with all signs and practical forms of solution.

It makes a thorough study of all fractional operations in problems of intermediate difficulty.

It masters tables and works easy problems in measures, weights, surfaces, contents, time.

Sight work with round numbers and easy parts of one dollar.

Multiplication and division of numbers ending in one or two ciphers.

Long division mastered with quotient always placed above dividend as preparation for decimal work.

Squares of easy numbers with pictures of same.

Master reading and writing numbers to billions.

Bank accounts as problems in addition and subtraction.

Add and subtract mixed numbers. Multiply whole by mixed numbers.

Comparison of values in solution of problems.

Bills from buying schedules.

Reduction of practical denominate number terms to higher and lower terms.

Pupils make problems; solve problems in which they secure their own data; as, walks, floors, window panes, bins, etc.; practical book problems, emphasizing short processes. Check and test results.

Complete statements of many problems without working.

Competitive speed drills.

Addition and subtraction of mixed numbers involving equalizing of denominators of fractions.

Fractional parts of numbers representing money, denominate expressions, concrete and abstract quantities.

Cubic contents—secure data; build in construction work; solve rational problems.

(To be Continued.)

NEW BOOKS.

An Elementary Treatise on Statics. By S. L. LONEY. Cambridge: The University Press; G. P. Putnam's Sons, American agents. Pp. 393. $4.00 net.

This work presupposes a knowledge of elementary calculus and solid geometry as well as the more fundamental notions of statics. Besides others it has chapters on Work, Center of Gravity, Machines, Attraction and Potential, and Graphic Solutions. The book seems to be written in Professor Loney's usual clear style and contains much to be commended. There are many examples covering a wide range of application and of varying degrees of difficulty.

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. HOBSON. Cambridge: The University Press; G. P. Putnam's Sons, American agents. Pp. 772. $6.50.

To adequately express the contents and nature of such an important and critical work as this in small space would be impossible. The arithmetic continuum, including the arithmetic theory of limits, and the nature of the functional relation form the basis for the theory, and the object is largely the finding of necessary and sufficient conditions for the validity of the limiting processes of analysis. The chapter headings are as follows: Number, Theory of Sets of Points, Transfinite Numbers and Order-Types, Functions of a Real Variable, Integration, Functions Defined by Sequences, Trigonometric Series.

The keen logic and ability of the author are apparent from beginning to end and whatever the topic one finds it treated by a master.

Kimball's Commercial Arithmetic. By GUSTAVUS S. KIMBALL. New York: G. P. Putnam's Sons. Pp. 418. $1.20 net.

This book is intended for use in normal, commercial and high schools, and for the higher grades of common schools. Though the book treats of many topics which are met only in the various forms of business, much stress is placed upon the fundamental processes.

The Teaching of Primary Arithmetic. By HENRY SUZZALLO. Boston: Houghton, Mifflin Company. Pp. 124. 60 cents.

The author has given in this little volume a rather clear and critical discussion of recent tendencies in methods of teaching arithmetic. The treatment is from the broad and modern standpoint, and is well worth reading by any teacher of elementary or secondary mathematics.

A Treatise on Analytical Geometry of Three Dimensions. By GEORGE SALMON. Fifth Edition, Revised by REGINALD A. P. ROGERS. Vol. I. New York: Longmans, Green and Company. Pp. 470. $3.00 net.

The many friends of Salmon's books will be delighted that a new edition of the solid geometry, which has been so long out of print, has made its appearance. The changes from the fourth edition have been mainly in the form of additions giving methods and points of view in harmony with modern usage. Some paragraphs in the nature of commentaries and a number of examples illustrative of the text have been added. The added material has improved what was already a splendid book and no doubt the new edition will receive a hearty welcome.

The New History. By JAMES H. ROBINSON. New York: The Macmillan Company. Pp. 266. $1.50 net.

The author desires to emphasize in this volume "the fact that history should not be regarded as a stationary subject which can only progress by refining its methods and accumulating, criticizing, and assimilating new material, but that it is bound to alter its ideals and aims with the general progress of society and of the social sciences, and that it should ultimately play an infinitely more important rôle in our intellectual life than it has hitherto done."

The layman as well as the historian will find it interesting reading.

The Philosophical Works of Descartes. Rendered into English by ELIZABETH S. HALDANE and G. R. T. ROSS. Cambridge: The University Press; G. P. Putnam's Sons, American agents. Vol. I. Pp. 452. $3.50.

This edition aims to give to the English reader all the works of Descartes which were originally intended for publication. Parts of his works had previously been translated but some of that had been long out of print and there never was sufficient to give a thorough comprehension of his views. The works translated in the volume are the "Rules," the "Method," the "Meditations" with the "Objections and Replies," part of the "Principles," the "Search after Truth," the "Passions," and the "Notes."

NOTES AND NEWS.

The spring meeting of the Philadelphia Section of the Association of Teachers of Mathematics was held on April 17. Interesting papers were read by Professor Bateman, of Bryn Mawr College, on "The Work of an English Mathematical Student," and by Dr. R. L. Moore, of the University of Pennsylvania, on "The Axioms of Geometry." The discussion on the latter was opened by Dr. Durell, of the Laurenceville School, and Dr. Rorer, of the William Penn High School.

The following officers for the ensuing year were elected: *President,* Professor Maurice J. Babb, University of Pennsylvania; *Vice-President,* Dr. Fletcher Durell, the Laurenceville School; *Secretary,* Elizabeth B. Albrecht, Philadelphia High School for Girls; *Members of the Executive Committee,* Professor C. L. Thornburg, Lehigh University, Mrs. Katherine D. Brown, Drexel Institute.

The Eighteenth Meeting.

The eighteenth meeting of the Association was held in Lyman Hall, Syracuse University, on Saturday, April 6, 1912. In the absence of Dr. Schwatt on account of illness, Dr. Metzler was asked to take the chair. The general subject of the meeting was the mathematical curriculum of the elementary school. The program of the morning was

1. Address:

   William H. Metzler, Dean of the Graduate School, Syracuse University.

2. What should comprise the subject matter of the Arithmetic in the Elementary Schools and in what grades should formal Arithmetic be taught?

   Speakers:

   H. J. Wightman, Superintendent of Schools, Lower Merion School District, Ardmore, Pa.

   Elmer E. Arnold, State Inspector of Schools, Albany, N. Y.

3. What minimum of practical applications should be included in a course in Arithmetic?

   What additional applications should be included for purposes of mental decipline?

SPEAKERS:

C. R. DOOLEY, in charge of the Educational Work of the Westinghouse Electric & Mfg. Co., Pittsburgh, Pa.

LOUIS L. PARK, Superintendent of Apprentices, American Locomotive Co., Schenectady, N. Y.

WILLIAM REED, National City Bank of New York.

After the morning session luncheon was served to those present by the University.

The programme for the afternoon was as follows:

4. MISCELLANEOUS BUSINESS.
5. In what operations in Arithmetic ought a pupil to be required to understand the reasons for the steps taken?

   In what operations should such a requirement be postponed to a maturer age?

SPEAKERS:

ARTHUR M. CURTIS, State Normal School, Oneonta, N. Y.

MISS M. ELSIE DAVIS, City Teachers' Training School, Buffalo, N. Y.

6. Should Algebra be taught in the Elementary Schools? If so, in what grades, and what should be the character of the work?

   Should Geometry be taught in the Elementary Schools? If so, in what grades, and to what extent?

SPEAKERS:

D. J. KELLY, Superintendent of Schools, Rome, N. Y.

WILLIAM F. LONG, Central High School, Pittsburgh, Pa.

By all those present the meeting was considered to be as good a one as we have ever had. Those not present at the meeting, though they will have the papers in THE TEACHER, missed a good deal of the finest enthusiasm. When such sentiments as "We need the data-finding problems to develop initiative," "The average pupil is a myth," "Arithmetic exists for the problems not the problems for arithmetic," etc., are the sentiments of earnest speakers then surely enthusiasm abounded.

H. T. HART,
Secretary.